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Algebra Manipulatives

Comparison and History

by Henri Picciotto

If you are not familiar with the basics of algebra manipulatives, I recommend you look at other articles on this site first, as this one is rather technical.

Article outline

Introduction

Even though they cannot make algebra easy, manipulatives can play an important role in the transition to a new algebra course:

- They provide access to symbol manipulation for students who had previously been frozen out of the course because of their weak number sense.
- They provide a geometric interpretation of symbol manipulation, thereby enriching all students' understanding, and making a powerful connection to another part of mathematics.
- They support cooperative learning, and help improve discourse in the algebra class by giving students objects to think with and talk about. It is in the context of such reflection and conversation that learning happens.

There are three main commercial versions of algebra manipulatives. In order of their appearance on the market, they are Algebra Tiles (Cuisenaire), the Lab Gear (Creative Publications), and Algeblocks (Southwestern Publishing). All three provide a worthwhile model of the distributive law. However, note that only the Lab Gear and Algeblocks allow work in three dimensions.

The Lab Gear

My creation! Read an introduction in:

- Early Mathematics and
- A New Algebra

Then, see the animated graphical mini-intro to the Lab Gear by George Collison of INTEC.

For a full presentation of the Lab Gear, see my algebra books, or the video course I helped write for Dr. Ed Dickey of the University of South Carolina. This article is adapted in part from the print material that accompanied the video. (How to get some of my publications.)

The Algebra Tiles

The Algebra Tiles are inexpensive and widespread. They make it possible to do most of the activities that are needed to introduce and explain the distributive law and factoring.

- The Algebra Tiles do not provide a corner piece. Since the corner piece is a very important support for beginners, I recommend you add corner pieces to your algebra tiles, either by purchasing them separately, or by just drawing them on paper.
- The absence of 3-D blocks is a limitation, but much can be done in two dimensions.
- Likewise, the fact that there is only one variable is a limitation, but it is not fatal.
- The lack of 5, 25, 5x, and 5y blocks does not really interfere with the key concepts, though it does make it inconvenient to represent expressions involving larger numbers.

Algeblocks

Algeblocks do have a variation on the corner piece (the quadrant mat) and they do support two variables and three dimensions. They lack the 5, 25, 5x, and 5y blocks, which is merely inconvenient. You can use them to do almost all the basic activities about the distributive law and factoring.

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Representation of Minus

The representation of minus is the most controversial part of manipulative models of polynomial algebra. The three commercially available products each handle it in a different way. These ways have consequences in four areas:

- the misconception that -x is negative, and x is positive
- the integrity of the area model of multiplication
- the representation of expressions such as 5x-(x-1)
- the complexity of the model

The **Algebra Tiles** model of minus is based on color: one color corresponds to positive numbers (blue, I believe), and another to negative numbers (black). This is then generalized to variables: blue x's represent x, and black x's represent -x.

- Advantage: the model is simple
- Disadvantages:
  - the model reinforces the misconception that -x is negative, and x is positive;
  - the area model of multiplication is not geometrically sound when minus is involved;
  - only the simplest expressions involving minus can be represented.

The **Algeblocks** model of minus is based on position: blocks in the shaded area of the various Algeblocks mats are taken to be preceded by a minus.

- Advantage: the model is simple
- Disadvantages:
  - the area model of multiplication is not geometrically sound when minus is involved;
  - only the simplest expressions involving minus can be represented.

The **Lab Gear** model involves two different representations, both based on position: blocks in the minus area of the workmat, and blocks sitting on top of other blocks ("upstairs") are taken to be preceded by a minus.

- Advantages:
  - the upstairs representation allows for a geometrically sound representation of minus in multiplication;
  - the two representations can be combined to represent expressions such as 5x-(x-1).
- Disadvantage: it is initially harder to learn than the other models.

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Multiplication with Minus

With the **Algebra Tiles**, to multiply (y+3)(y-2), you do the same thing that you would do with (y+3)(y+2), except that you use the "minus" colored tiles for -2, and for -2y and -6 in the product.

Similarly for (y-3)(y-2), you do the same thing that you would do with (y+3)(y+2), except that you use the "minus" colored blocks for -2, and for -2y and -3y in the product.

Note that in this model, the products (y+3)(y+2), (y+3)(y-2), (y-3)(y+2), and (y-3)(y-2), are all congruent, which is geometrically incorrect, since for example y+3 clearly should be longer than y-3.

With **Algeblocks**, factor blocks are placed on the right or above the center of the quadrant mat if they are preceded by a plus, and to the left or below, if they are preceded by a minus. The resulting product subrectangles are considered to be preceded by a plus for the 1st and 3rd quadrant, or a minus for the 2nd and 4th quadrant.

In this model also, the area model loses its geometric integrity when minus is used:

- The factors are sometimes embedded in the product rectangle, distorting its dimensions.
- Once again, the products (y+3)(y+2), (y+3)(y-2), (y-3)(y+2), and (y-3)(y-2), are all congruent, which is geometrically incorrect.

With the **Lab Gear**, the upstairs representation of minus accurately represents y-2 as 2 units shorter than

y. It is then possible to multiply in the corner piece, and use upstairs blocks in the product, to obtain rectangles with geometrically correct dimensions. However how to do this is not easy to learn in a case such as (y-3)(y-2), and many Lab Gear users stop short of teaching this to their students. This is not a big problem, because the Lab Gear materials encourage a transition from blocks to symbols with the help of the "multiplication table" format for polynomial multiplication. That format is visually related to the basic Lab Gear multiplication format, it works fine with minus, and it guarantees that the student does not remain dependent on the blocks.

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Curriculum Materials

In fact, I encourage you to compare the curriculum materials provided with the three models. In my opinion, the Algebra Tiles manual is confusing and essentially unusable. There is much of value in the Algeblocks binder, which seems in large part inspired by the original Lab Gear books. Moreover, their ingenious (if mathematically flawed) approach to minus does make the Algeblocks easier to use. But for mathematical correctness, depth, and range, check out my *Lab Gear Activities for Algebra 1*. The binder is the result of many years of work with algebra manipulatives, with both teachers and students, and includes all the basic lessons on the distributive law, a rich introduction to equation solving, plus three important features not found in either of the other programs:

- explicit attention to the transition from blocks to symbols
- connections to the graphing of linear and quadratic functions
- a remarkably accessible approach to completing the square and the quadratic formula

For extended work on integer arithmetic, plus a general introduction to Lab Gear at a more basic level, see *The Algebra Lab: Middle School*. (How to get some of my publications.)

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History of Algebra Manipulatives

The first use of manipulatives to illustrate algebraic ideas was by math educator Zoltan Dienes, who used base-10 blocks. Using the "rod" (10) as x, and the "flat" (100) as x^2, he showed how to use base 10 blocks to illustrate the distributive law. For example, $(x+5)(x+2)=x^2+7x+10$ can be seen by making a 12 by 15 rectangle, and seeing that its area is $100+7\cdot10+10$. The idea was powerful, and launched the idea of algebra manipulatives, but this model falls short when trying to factor: $x^2+7x+10$, represented by $100+7\cdot10+10$ can be arranged into two rectangles: 12 by 15, or 10 by 18. The first one is the correct factoring $(x+2)(x+5)$, but the second is not: $x(x+8)$ does not equal $x^2+7x+10$.

Mary Laycock improved on Dienes' model, by using multi-base blocks. Instead of just working with base 10 blocks, a trinomial factoring had to work in all bases. In other words, the same layout should work whether x=3, 5, 6, or 10. She also introduced the "upstairs" representation of minus, which made it possible to represent a product involving minus like (x-1)(x+1) in a geometically correct way.

Peter Rasmussen used base ten tiles, plus 5-, 25-, and 50-tiles for convenience. More importantly, he created the non-commensurable x, which solved the problem of false factorings encountered when using arithmetic blocks for variables. He also laid out multiplications in a tray that was a precursor to the corner piece. His MathTiles were placed along the tray's frame in such a way that the student saw them edge-on, thereby suggesting the one-dimensionality of linear measurements of the dimensions of the two-dimensional rectangles inside the tray. His model of minus combined Laycock's upstairs method with a color scheme. The tiles were only painted on one side, so that if a tile is turned over to its unpainted side, it is considered negative. (Number tiles, x-tiles, and x^2-tiles each had their own color.)

The Algebra Tiles were based on Rasmussen's ground-breaking model, without the upstairs representation of minus. Unfortunately, they dropped the multiplication tray. Fortunately, they did keep the non-commensurable x.

The Lab Gear was based on Rasmussen's and Laycock's models, keeping the non-commensurable x, and adding a non-commensurable y, and blocks for 5x, 5y, and 25, for convenience. (Blocks for 10, 100, and 1000 can be readily added to the Lab Gear by using base 10 blocks, which could be quite useful in younger grades.) The corner piece extends the uses of the tray for multiplication into the third dimension, and 3D multiplication is supported by x^3, x^2y, y^2x, and y^3 blocks. The minus model is extended beyond the upstairs model by the use of the workmat, with its minus area. The workmat is an environment for working with polynomials involving minus signs, as well as for equation solving and inequalities.

Algeblocks are based on the Lab Gear, including the x and y variables, and variations on the workmat. However, the 5, 25, 5x, 5y, are missing. The corner piece is replaced by the quadrant mat, and the upstairs representation of minus is abandoned.

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Teacher Resource Page

Add & Subtract Signed Numbers with Algebra Tiles

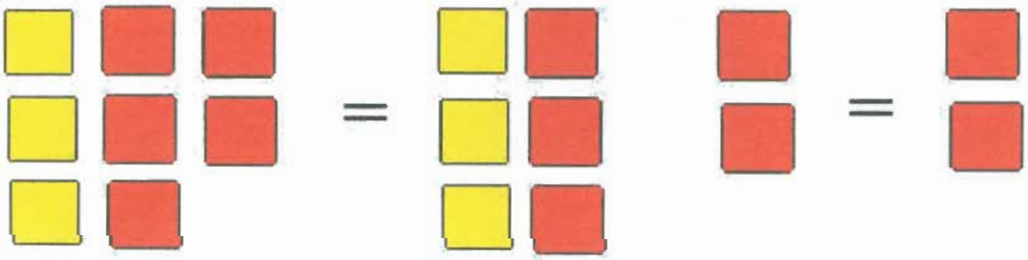
Math A

See Homemade Algebra Tiles
for template for making tiles for your students to use in class.

Key:  =1  =-1

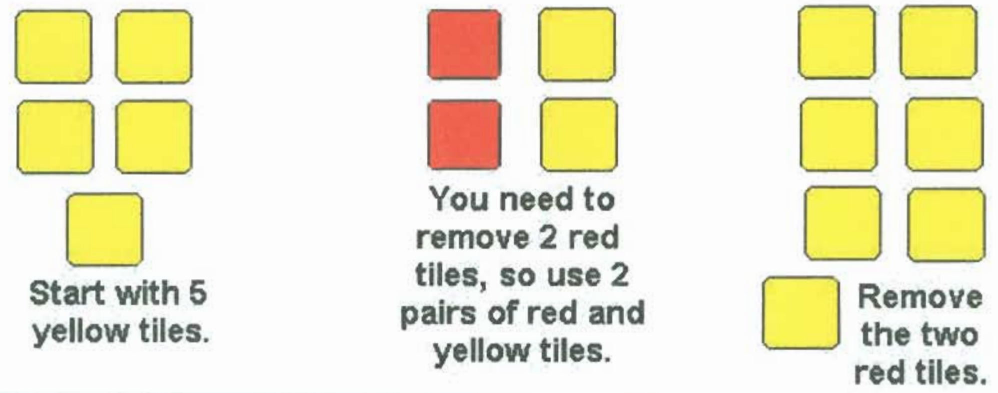
*Be sure to stress that a red tile and a yellow tile (of the same size) will cancel each other out (will add to 0) and will be removed from the table.

1. Addition: Show 3 + (-5) = -2



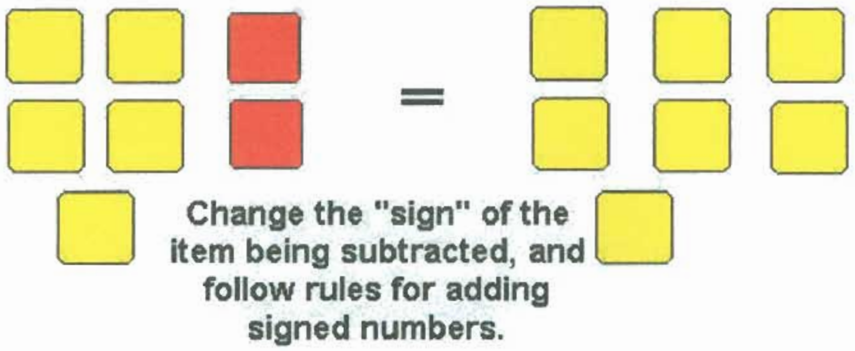
2. Subtraction: Show 5 - (-2) = 7

The concept of "subtraction" is the removal of a certain number of tiles. In this problem it is necessary to remove 2 negative (red) tiles. Since positive five contains NO negative tiles to be removed, it is necessary to introduce the negative (red) tiles. Notice the middle step of adding 0 (one red and one yellow) to the problem twice.



Or explain as:

A simpler way to explain this process, and to reinforce the "rules" for subtracting signed numbers, is to start with five positive tiles (yellow) and two negative tiles (red) and then instruct the students to "change the sign" of the tiles being subtracted by replacing them with positive yellow tiles.



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Roberts