PAPER-FOLDING CONSTRUCTIONS

Most of the constructions of Euclidean Geometry can be done by folding paper. Ordinary kitchen waxed paper is excellent for this because it is possible to see lines through it. The following is a list of constructions you should try. They are listed in order of increasing difficulty.

1. Mark two points on your paper. Label them points P and Q. Construct a line through P and Q.
2. Construct a line perpendicular to your line $\overleftrightarrow{PQ}$.
3. On another sheet of paper, fold a line m. Construct a perpendicular to line m through a point P on line m. (Select point P before making the perpendicular, of course.)
4. Select a point Q not on line m. Construct a perpendicular to line m through point Q.
5. On another sheet of paper, fold a line. Construct a perpendicular bisector to line segment $\overline{PQ}$ on that line. (Of course, select points P and Q prior to constructing the bisector.)
6. On another sheet of paper, fold a line. Now construct a line which is parallel to that line.
7. Select a point P not on either of the two parallel lines. Construct a third parallel line that passes through point P.
8. On another sheet of paper, fold an angle and label it $\angle$AOB. Construct the bisector of $\angle$AOB.
9. Using another sheet of paper, find two ways to construct a right angle.
10. Construct a 45-degree angle. (Use another sheet of paper when needed.)
11. Construct a 30-degree angle.
12. Fold a triangle. Show the sum of the measures of the interior angles of the triangle is 180 degrees. You may find it easier to cut out the triangle.
13. Construct a right triangle.
14. Construct a right, isosceles triangle.
15. Construct a right triangle where one leg is twice the measure of the other leg.
16. Construct a 30-60-90 triangle
17. Trisect a line.
18. On a new sheet of paper, fold a large triangle of any type (equilateral, isosceles, or scalene; acute, right, or obtuse). Label this triangle $\triangle$ABC. Construct all of the following on this triangle:

    Construct the three medians.
    Construct the three altitudes.
    Construct the three angle bisectors.
    Construct the three, perpendicular side bisectors.
    DISCUSS: circumcenter, incenter, orthocenter, and centroid.
    RESEARCH: 9-point or Feuerbach circle and Euler's line.

19. Other constructions: diagonals of quadrilaterals, transversals, vertical angles, and altitudes of quadrilaterals.

- Paper can be folded so that a straight line can be superimposed on another straight line on the same sheet.
- Lines and angles are said to be equal if they coincide when one can be superimposed upon another by folding the paper.

If these assumptions are accepted, then it is possible to perform all the ıstructions of plane Euclidean geometry by folding and creasing.

Patterns for folding a great variety of polyhedra will be found in the lowing publications:

HARTLEY, MILES C. *Patterns of Polyhedrons*. Chicago: The author, University of Illinois, 1945.

CUNDY, H. M. and ROLLETT, A. P. *Mathematical Models*. London: Oxford University Press, 1952.

The writer wishes to give credit to those who have previously described ny of the paper-folding projects explained above. The writer is most in-ıted to Robert C. Yates who furnished the original inspiration and information for using these materials.

References on paper folding:

BERGER, EMIL; JOSEPH, MARGARET; SAUPE, ETHEL; and UTH, CARL. "Devices for the Mathematics Laboratory." *The Mathematics Teacher* 44:247-49; 48:42-44, 247, 49.

LEEMING, JOSEPH. *Fun with Paper*. Philadelphia: J. P. Lippincott Co., 1939.

ROW, SUNDARA. *Geometric Exercises in Paper Folding*. Chicago: The Open Court Publishing Co., 1941.

YATES, ROBERT C. *Geometrical Tools*. St. Louis: Educational Publishers, 1949.

All figures for this manuscript were drawn by Charles B. Bastis, University ·h School, Minneapolis, Minnesota.

How To Fold the Basic Constructions

A variety of geometric figures and relationships can be demonstrated by following the directions below. If you have a supply of wax paper, we are all set for a new way of learning mathematics.

1. Folding a straight line

Any point P of one portion of the sheet of paper is folded over and held coincident with any point Q of the other portion. While these points are held together tightly by the thumb and a finger of one hand, the fold is creased with the thumb and a finger of the other hand. This crease forms at points equidistant from P and Q. The crease is extended by holding the crease tightly with the thumb and finger of both hands, then pulling the hands apart. The tension used in completing the crease should be kept constant on both surfaces. Thus the distance from points P and Q to the crease remains equal on each portion of the sheet. The crease formed is then the locus of all points of the sheet which are equidistant from P and Q. Is this locus a straight line?

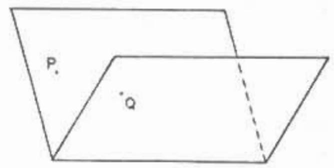


Figure 1

2. A straight line through a given point

Fold the sheet over with the given point on the outside. Carefully form a short crease that passes through the point. Extend the crease as described above.

3. A line perpendicular to a given straight line

Fold the sheet over so that a segment of the given line AB is folded over on itself. Hold the lines together tightly with the thumb and finger of both hands. Form the crease by pulling the hands apart with the right thumb and finger sliding. Why is the straight angle formed by the given line AB bisected by the crease CD?

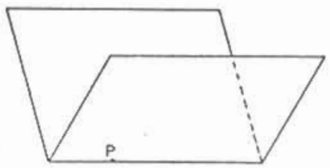


Figure 2

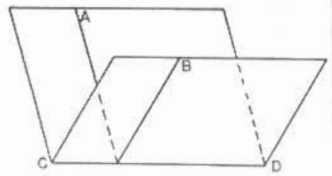


Figure 3

4. The perpendicular to a line at a point on the line

Fold the paper so the given line AB is superimposed on itself as in Number 3 above and so that the crease passes through the given point P.

Fold B on A but before creasing slide the paper, keeping the line coincident with itself, until the crease will pass through the given point P. Why is the fold through P perpendicular to AB?

5. A line perpendicular to a given line and passing through a given point P not on the line

This construction uses the same method of folding as Number 4 above.

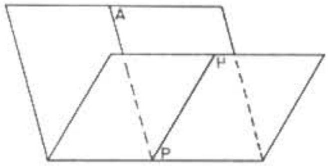


Figure 4

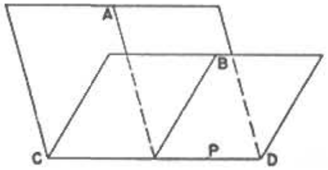


Figure 5

6. The perpendicular bisector of a given line segment

Fold the paper so that the end points of the given line AB are superimposed on each other. Why is this crease CD the perpendicular bisector of AB? Locate any point on the perpendicular bisector. Test by superposition to see if this point is equally distant from A and B.

7. A line parallel to a given straight line

First fold the perpendicular EF to the given line AB as in Number 3. Next fold a perpendicular to EF. Why is this last crease CD parallel to the given line AB?

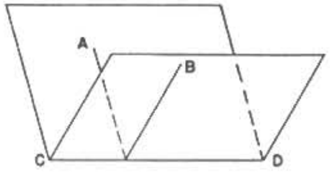


Figure 6

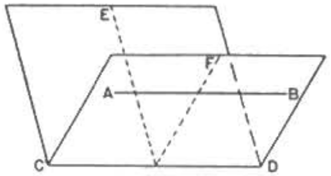


Figure 7

8. A line through a given point and parallel to a given straight line

First fold a line CD through the given point P perpendicular to the given line AB as in Number 5. In a similar way fold a line EF through the given

point P and perpendicular to the crease CD formed by the first fold. Why does this crease provide the required line?

9. The bisector of a given angle

Fold the paper so that the terminal sides AC and BC of the given angle ACB coincide. Why does the crease pass through the vertex and divide the given angle into two equal angles?

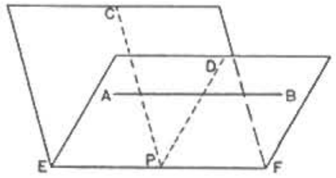


Figure 8

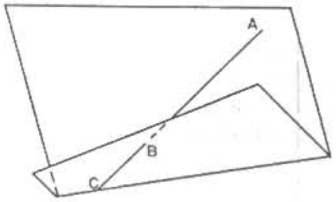


Figure 9

10. The location of equally spaced points along a line CD

Establish any convenient length as the unit length by folding a segment of the line upon itself. Form several equal and parallel folds by folding back and forth and creasing to form folds similar to those of an accordion.

11. The formation of a right angle

Take any piece of paper, one edge ($b\ x\ c$) of which must be straight as in Figure 11A below. Fold one end down at an acute angle as shown in Figure 11B. Then fold b over to touch c, making Figure 11C. Why is the angle YXZ a right angle?

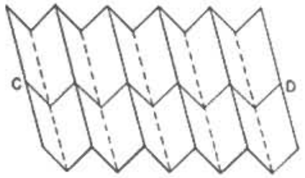


Figure 10

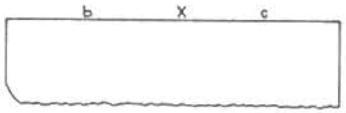


Figure 11A

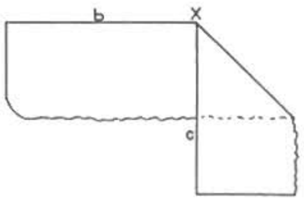


Figure 11B

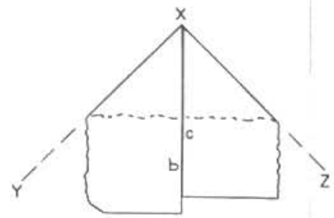


Figure 11C

Geometric Concepts Illustrated by Paper Folding

12. Vertical angles

Fold any two intersecting creases AB and CD intersecting at O. Compare the vertical angles by folding through the vertex O, placing BO on CO. Do AO and DO coincide? Are vertical angles equal?

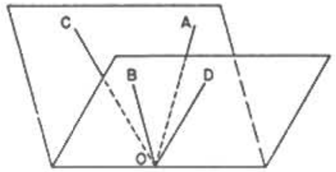


Figure 12

13. The sum of the angles of a triangle

(a) Fold the altitude BD of the given triangle ABC (Figure 13A).

(b) Fold the vertex of the triangle B upon the base of the altitude D (Figure 13B).

(c) Fold the base angle vertices A and C to the base of the altitude D. Does $\angle A + \angle B + \angle C$ make up a straight angle (Figure 13C)?

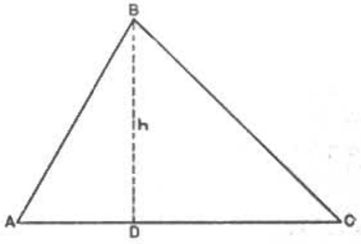


Figure 13A

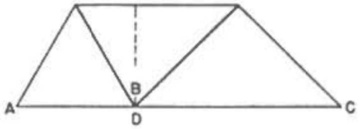


Figure 13B

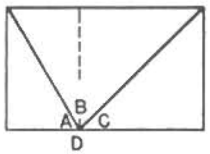


Figure 13C

14. The area of a triangle

In Figure 13C the rectangular shape has sides equal to one-half the base of triangle ABC and one-half the altitude BD (Figure 14). What is the area of the rectangle? How does the area of the original triangle ABC compare with this rectangle? What then is the area of the triangle?

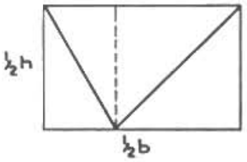


Figure 14

15. The midpoint of the hypotenuse of a right triangle

(a) Fold or draw any right triangle ABC (Figure 15A).

(b) Bisect the hypotenuse AB by folding A on B. Fold the line from the midpoint D to C (Figure 15B).

(c) Compare CD, AD, and BD by folding a crease through D. Will CD and BD coincide? Fold another crease through D to see if CD and AD will coincide. Is $CD = AD = BD$?

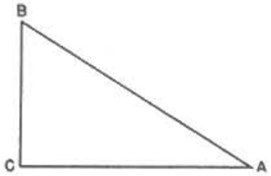


Figure 15A

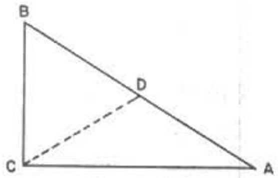


Figure 15B

16. The base angles of an isosceles triangle

Fold the perpendicular bisector BD of a given line segment AC. Crease oblique lines AB and BC from the ends of the given line to a common point B on the perpendicular bisector to form an isosceles triangle ABC. Compare the base angles by superposition by folding along BD. Are angles A and C equal?

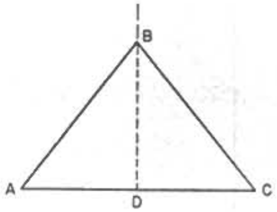


Figure 16

17. The intersection of the altitudes of a triangle

Fold the altitudes to each side of the given triangle. Do they intersect in a common point? What is the intersection point of two altitudes called? How do the distances from the point of intersection of these altitudes to the vertices and the bases of the triangle compare?

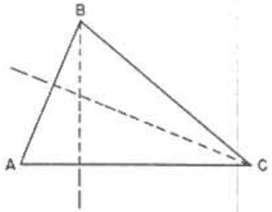


Figure 17

18. The intersection of the angle bisectors of a triangle

Fold the bisectors of each angle of the given triangle. Do the bisectors intersect in a common point? What is the point of intersection of two angle bisectors called? Fold the perpendicular from this point of intersection of two angle bisectors to each side of the triangle. Compare the lengths of these perpendiculars by superposition. Are the lengths equal?

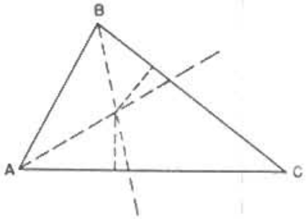


Figure 18

19. The intersection of the perpendicular bisectors of the sides of a triangle

Fold the perpendicular bisectors of each side of the given acute triangle. What is the common point of intersection of these lines called? Fold creases from this point to each vertex of the triangle. Compare these lengths by superposition. Are these lengths equal?

20. The intersection of the medians of a triangle

Bisect the three sides of the given triangle. Fold the crease from the midpoint of each side to the opposite vertex. What is this common point of intersection called? How do the distances from the point of intersection of two medians to each vertex of the triangle compare? Try balancing the triangle by placing it on a pin at the intersection of two medians. What is this point called?

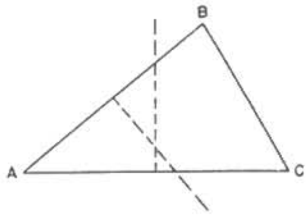


Figure 19

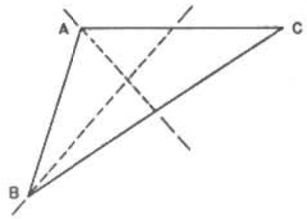


Figure 20

21. The area of a parallelogram

Cut a trapezoid with one side CB perpendicular to the parallel sides. Fold the altitude DE. Fold CF parallel to AD. When triangle FCB is folded back, $ADCF$ is a parallelogram. When triangle ADE is folded back, $DCBE$ is a rectangle. Are triangles ADE and FCB congruent? Is rectangle $BCDE$ equal to parallelogram $ADCF$? What is the formula for the area of a parallelogram?

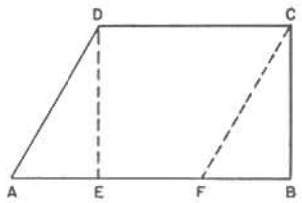


Figure 21

22. The square on the hypotenuse is equal to the sum of the squares on the two other legs of a right triangle

Use a given square $ABCD$. Make any crease GH. Complete the square $GHEF$ by forming right angles at G and H. Fold GJ, HK, EL, and FM by folds perpendicular to the sides of the given square $ABCD$. Note that $LH = GK = FJ = AF$. How can you prove that $h^2 + g^2 = c^2$?

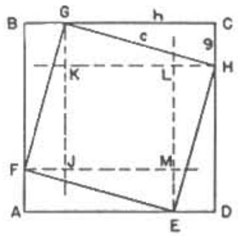


Figure 22

23. The diagonals of a parallelogram

Fold the diagonals of a given parallelogram. Compare the lengths of intersected segments by superposition. Are the diagonals of a parallelogram equal? Do the diagonals bisect each other?

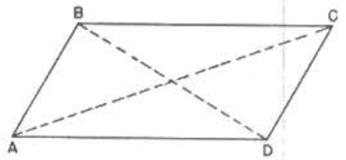


Figure 23

24. The median of a trapezoid

Fold the altitudes at both ends of the shorter base CI and DJ of the trapezoid $ABCD$. Bisect each nonparallel side and connect these midpoints with a crease EF. Does this median EF bisect the altitudes? Is this median EF perpendicular to the altitudes? Is this median parallel to the bases? Fold A on I and B on J. How does the sum of CD and AB compare with the median EF?

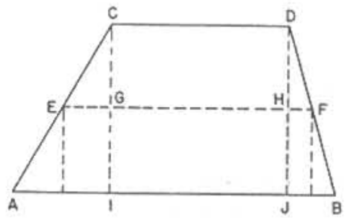


Figure 24

25. The diagonals of a rhombus

Fold the diagonals of a given rhombus $ABDC$. Compare angles and lengths of the diagonals by superposition. Do the diagonals intersect at right angles? Do the diagonals bisect each other? Is triangle ABC congruent to triangle BCD? What area will be found by the product of AD and CB?

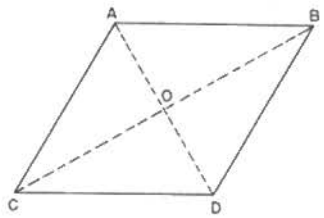


Figure 25

26. A line midway between the base and vertex of a triangle bisects the altitude and equals one-half the length of the base

Bisect two sides of the triangle ABC. Fold a crease through these mid-points EF. Fold the altitude to the side which is not bisected. Is EF the perpendicular bisector of BD? Is EF parallel to AC? Fold A and C upon D. How does the length of EF compare with the length of AC?

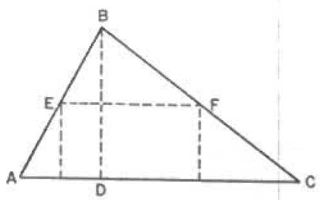


Figure 26

Activity: Concurrency Theorems

Materials Needed: Wax paper or scrap paper, straightedge, compass, pencil.

Directions:

1. Using a large sheet of paper, draw a large acute, scalene triangle. Label the vertices A, B and C.

2. Locate the midpoint of each side by folding. Label the midpoints 1, 2, and 3.

3. Fold all three altitudes. To fold the altitude from vertex C, you need to fold the paper so that side $\overline{AB}$ is superimposed on itself and the crease passes through vertex C. Fold the altitudes from vertex A and B. Label the point where these altitudes meet the sides of the triangle 4, 5 and 6. Label the point of intersection of the altitudes 0.

4. Locate the midpoint of $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ by folding. Label these points 7, 8 and 9.

5. Using your compass, try to draw a circle through all nine points (1,2,3,...,9). Determining the center of the circle and the radius is your challenge!

The circle you have found is called the <u>Nine-Point Circle</u>, or <u>Feuerbach Circle</u>.

Additional Questions:

1. What would happen if you started with an equilateral triangle?

2. What would happen if you started with an obtuse triangle? (Hint - think about the altitudes and point 0.)

3. What is the definition of the following words?

   a. Circumcenter
   b. Incenter
   c. Centroid
   d. Orthocenter

4. What is the Euler Line? (You will need to visit the library.)

Activity: Concurrency Theorems

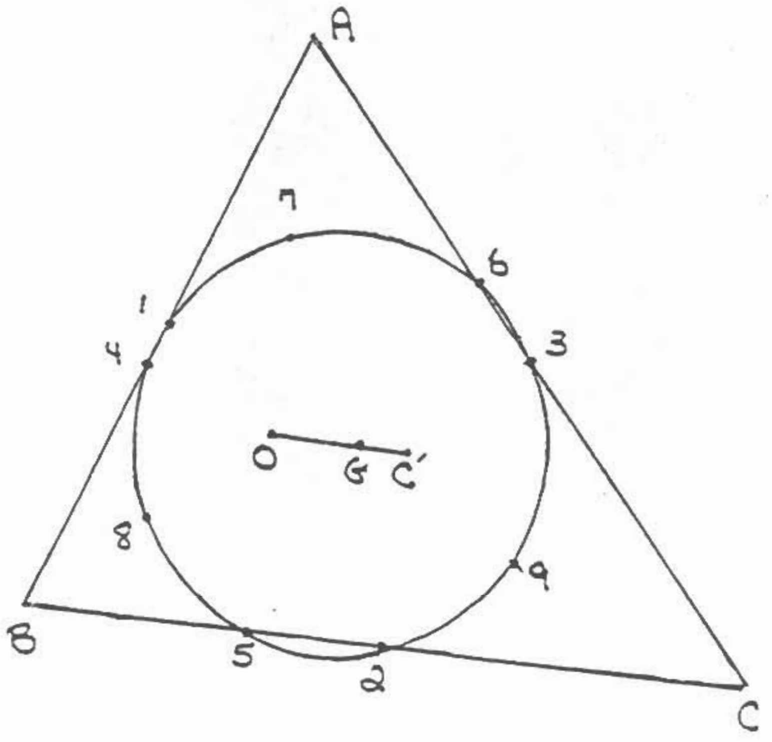
Definitions:

1. The point of concurrency of the perpendicular bisectors of a triangle is called the circumcenter. The circumcenter is equidistant from the vertices of the triangle and is the center of the circumscribed circle.

2. The point of concurrency of the angle bisectors of a triangle is called the incenter. The incenter is equidistant from the sides of the triangle and is the center of the inscribed circle.

3. The point of concurrency of the medians of a triangle is called the centroid. The centroid is two thirds the way from each vertex to the midpoint of the opposite side. The centroid is also called the center of gravity.

4. The point of concurrency of the altitudes (or the lines containing the altitudes) of a triangle is called the orthocenter. In an obtuse triangle, the orthocenter will lie in the exterior of the triangle.



5. The orthocenter - O, centroid - G and circumcenter - C′ of a triangle are collinear. This line is referred to as Euler's Line. The endpoints are O and C and |OG| = 2|GC| (i.e. the centroid is two thirds the way from the orthocenter to the circumcenter).

Activity: Concurrency Theorems

Additional Notes

1. The center of the Nine-Point Circle lies on the Euler Line, halfway between the orthocenter and circumcenter of the triangle. The radius of this circle is half the radius of the circumscribed circle.

2. The Nine-Point Circle, Euler or Feuerbach make excellent topics for a project, research paper or oral presentation.

3. Resources:

   a. Paper Folding for the Mathematics Class. Donovan Johnson, NCTM, 1972.

   b. Geometry. Isidore Dressler, Amsco School Publications, 1973.